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Hydrodynamic model for surface nematic viscosity

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We propose a hydrodynamic model that justifies the surface viscosity already introduced as a phenomenological parameter in the surface dynamics of nematic liquid crystals. In this model the surface orienting field of a substrate is assumed to be *diluted* in a thin boundary layer that is described by the balance equations of the Ericksen-Leslie theory. A two-dimensional description is then recovered from this three-dimensional model: it indicates the premises on which a general theory of surface dynamics could be developed.
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I. INTRODUCTION

Ordered systems exhibit interesting surface properties, some of which have been extensively studied, mostly in statics, such as wetting and surface segregation. In oriented systems, such as magnets and liquid crystals, the orientation also comes into play and the statics of what is usually called the *anchoring* is now fairly well understood, at least for nematic liquid crystals.

In a nematic liquid crystal, when the surface orientation is forced away from its spontaneous easy direction, the new equilibrium configuration is classically described by introducing an anchoring surface energy. The *statics* is then described as the equilibrium of the torque $\boldsymbol{\mathcal{K}}_s$ produced by the surface and the torque $\boldsymbol{\mathcal{K}}_a$ exerted by the disturbing action:

$$\boldsymbol{\mathcal{K}}_s+\boldsymbol{\mathcal{K}}_a=\mathbf{0}.$$

Measurements of these torques have made it possible to construct the surface potential. Regardless of its dependence on the orientation, its strength is usually written as *K/L*, where *K* is a typical curvature elastic constant and *L* is the anchoring *extrapolation length*: With $K=10^{-6}$ dyn, *L* is found in the range of a few hundred angstroms for strong anchorings up to a few micrometers for very weak anchorings.

The question of how to describe the anchoring *dynamics* was indeed posed rather early: It was solved by introducing a phenomenological *surface* viscosity $\ell\eta$, where ℓ is a length and η a bulk viscosity. The phenomenological dynamic equation for a surface was written in the form

$$\ell\eta\,\boldsymbol{n}_s\times\dot{\boldsymbol{n}}_s=\boldsymbol{\mathcal{K}}_s+\boldsymbol{\mathcal{K}}_a, \tag{1}$$

where $\dot{\boldsymbol{n}}_s$ is the time derivative of the surface director $\boldsymbol{n}_s$ (see, for instance, [1] and [2]).

Many further questions could still be asked. What is the dissipation mechanism behind the surface viscosity? What is the order of magnitude of ℓ? Is ℓ comparable to *L*, as conjectured in [3], or are these lengths completely independent? In this paper we attempt to answer these questions by proposing an explicit hydrodynamic model for both the structure and the dynamics of a surface nematic layer. This model relies on a three-dimensional description of the surface anchoring effects, which are thought of as *diluted* in a thin boundary layer. Such a three-dimensional picture of the anchoring could be judged hazardous because we apply the macroscopic theory on a scale not necessarily large enough relative to the molecular scale. We will show, however, that a genuine two-dimensional model can actually be retrieved from it by integrating the evolution equations across the boundary layer: We shall recover the phenomenological equation (1), which thus becomes fully justified.

Assuming that η is comparable to the bulk rotational viscosity γ_1, we obtain from recent measurements of the surface viscosity [4] that ℓ can be estimated to be in the range of 10^2 Å. With this estimate, by a purely dimensional analysis of Eq. (1), we see that there should be a *surface* relaxation time τ_s, which can be estimated by

$$\frac{1}{\tau_s}\sim\frac{K}{\ell L\eta},$$

and so it approximately falls into the range 1–10 μsec. Since this is a very short time for liquid crystal dynamics, to observe any possible effect so fast we must use a perturbation with a frequency comparable to $1/\tau_s$. In practice, it is usually more convenient to distort first the liquid crystal into an equilibrium configuration by means of a static electric field, which is then suddenly removed, leaving the torques unbalanced: Thus the distorted configuration freely relaxes. We will show, however, that in this setting, which is special though easily obtained in practice, the actual relaxation rate $1/\tau$ is slower than $1/\tau_s$:

$$\frac{1}{\tau}\sim\frac{1}{\tau_s}\,\frac{\ell}{L},$$

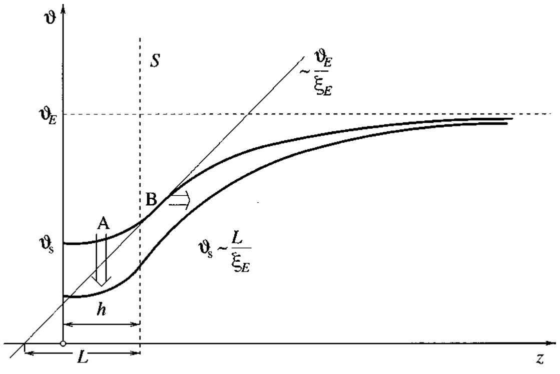

FIG. 1. Structure and dynamics of the boundary layer. In this three-dimensional model, the surface orienting field is *diluted* in a boundary layer with thickness h. The dashed line represents the ideal plane S at the interface between the boundary layer and the bulk. The surface torques across S are those described in the usual two-dimensional model. The solid curve represents the distorting effect in the bulk due to an external electric field with coherence length ξ_E. The easy anchoring of the plate is $\vartheta=0$; ϑ_E is the orientation forced in the bulk by the electric field. The slope of the solid curve at $z=h$ is $\sim \vartheta_E/\xi_E$. The surface orientation ϑ_s is supposed to be small and so the extrapolation length L is easily related to it: $\vartheta_s \sim L/\xi_E$.

and so this relaxation is governed by the friction with the bulk. In the more general case, there are physically admissible distortions that start relaxing at the faster rate $1/\tau_s$.

The plan of the paper is as follows. In Sec. II we describe the model for a single three-dimensional layer, where the surface effects are diluted. In Sec. III we recall the classical Ericksen-Leslie theory to be applied in Sec. IV. In Sec. V we describe the static structure of the boundary layer and in Sec. VI its relaxation dynamics. Finally, in Sec. VII we summarize the main conclusions of this paper and the avenues for further work that it opens.

II. LAYER MODEL

Think of a nematic liquid crystal that occupies the half space delimited by a plate (see Fig. 1). Imagine that the natural equilibrium configuration for the director field $\boldsymbol{n}$ has been distorted, say, by applying an electric field, which was strong enough to break the anchoring on the supporting plate. Then the field is removed and the system starts relaxing freely towards equilibrium.

Here we study the onset in this relaxation process: It is the *incipient* motion of $\boldsymbol{n}$ in the vicinity of the supporting plate, just after the perturbing cause has been removed. In the vicinity of the plate, the director field is expected to relax under the effect of the restoring torques exerted by the anchoring substrate. Such a surface effect is thought of as *diluted* in a *thin* boundary layer, whose thickness h is much smaller than the *extrapolation* length L associated with the anchoring on the plate. We also take h to be larger than the molecular scale so that the classic three-dimensional continuum theory could still be applied within the boundary layer.

The motion of the director entrains a hydrodynamic flow. After a short transient from the relaxation onset, a velocity field $\boldsymbol{v}$ is established in the half space, whose vorticity is mainly concentrated near the plate. This is indeed the main feature of our model: the vorticity of the incipient velocity essentially vanishes very close to the boundary. Thus this model cannot be applied long after the flow starts because the vorticity soon diffuses in the whole region occupied by the body. In some practical applications, however, when a stable equilibrium configuration is being perturbed that could coexist with others with the same elastic free energy, the initial velocity of $\boldsymbol{n}$ might indeed tell which of the other local minimizers will be approached after the perturbation ceases. Thus, without actually solving the dynamic equations for both $\boldsymbol{n}$ and $\boldsymbol{v}$ during the whole time evolution, we can say in what energy well the director will eventually be trapped again. Bistable anchorings, for example, provide ideal applications of our model.

In the next section, following both [5] and [6], we review the essential features of the Ericksen-Leslie theory in a form that is appropriate wherever a boundary layer embodies a surface effect in the dynamics of nematic liquid crystals.

III. BALANCE EQUATIONS

Here we assume that no body force acts on the liquid crystal and that the inertia associated with its motion is negligible. Under these assumptions, the balance laws for forces and torques can be given the point form

$$\operatorname{div} \mathbf{T}=\mathbf{0}, \tag{2}$$

$$\boldsymbol{k}+\boldsymbol{K}_s=\mathbf{0}, \tag{3}$$

where $\mathbf{T}$ is the *stress tensor*, $\boldsymbol{k}$ is the *internal body torque* per unit volume, and $\boldsymbol{K}_s$ is the *diluted surface torque* per unit volume.

Both fields $\mathbf{T}$ and $\boldsymbol{k}$ comprise a static and a dynamic component: Their constitutive laws, which are now classic, are recalled below. $\boldsymbol{K}_s$ reflects our way of treating surface effects: It is a field that rapidly decays away from the supporting plate and it already vanishes just beyond the height h. It could easily be related to a surface potential provided this is also thought of as diluted in the boundary layer.

Let $\mathcal{B}$ be any region in space occupied by the liquid crystal. $\boldsymbol{K}_s$ can be derived from the potential W_s whenever the power it expends can be classically expressed as the time derivative of the integral of W_s over $\mathcal{B}$:

$$\int_{\mathcal{B}} \boldsymbol{w}\cdot\boldsymbol{K}_s=\frac{d}{dt}\left(\int_{\mathcal{B}} W_s(\boldsymbol{n})\right), \tag{4}$$

where $\boldsymbol{w}$ is the absolute angular velocity of $\boldsymbol{n}$ defined by

$$\dot{\boldsymbol{n}}=\boldsymbol{w}\times\boldsymbol{n}. \tag{5}$$

It follows from Eqs. (4) and (5) that

$$\int_{\mathcal{B}} \boldsymbol{w}\cdot\boldsymbol{K}_s=\int_{\mathcal{B}} \frac{\partial W_s}{\partial \boldsymbol{n}}\cdot\dot{\boldsymbol{n}}=\int_{\mathcal{B}} \boldsymbol{w}\cdot\boldsymbol{n}\times\frac{\partial W_s}{\partial \boldsymbol{n}}, \tag{6}$$

which is valid for every choice of $\boldsymbol{w}$. Thus $\boldsymbol{K}_s$ can be written as

$$\boldsymbol{K}_s=\boldsymbol{n}\times\boldsymbol{H}_s, \tag{7}$$

where

$$\boldsymbol{H}_s:=\frac{\partial W_s}{\partial \boldsymbol{n}} \tag{8}$$

is the *molecular field* induced by the surface. $\boldsymbol{H}_s$ is everywhere orthogonal to $\boldsymbol{n}$ and so it will be represented in the form

$$\boldsymbol{H}_s=H_s\boldsymbol{n}_\perp,$$

with $\boldsymbol{n}_\perp$ a unit vector orthogonal to $\boldsymbol{n}$.

The stress tensor $\mathbf{T}$ can be split in the *elastic* stress tensor $\mathbf{T}^{(e)}$ and the *viscous* stress tensor $\mathbf{T}^{(v)}$. They obey the constitutive laws

$$\mathbf{T}^{(e)}=-p\mathbf{I}-(\nabla\boldsymbol{n})^T\frac{\partial W}{\partial\nabla\boldsymbol{n}}, \tag{9}$$

$$\begin{aligned}\mathbf{T}^{(v)}=&\,\alpha_1(\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n}+\alpha_2\boldsymbol{\omega}\times\boldsymbol{n}\otimes\boldsymbol{n}+\alpha_3\boldsymbol{n}\otimes\boldsymbol{\omega}\times\boldsymbol{n}+\alpha_4\mathbf{D}\\&+\alpha_5\mathbf{D}\boldsymbol{n}\otimes\boldsymbol{n}+\alpha_6\boldsymbol{n}\otimes\mathbf{D}\boldsymbol{n},\end{aligned} \tag{10}$$

where p is a hydrostatic pressure raised by the incompressibility constraint, $W=W(\boldsymbol{n},\nabla\boldsymbol{n})$ is the elastic free energy per unit volume, α_1–α_6 are Leslie's viscosity coefficients, $\mathbf{D}$ is the shear measure

$$\mathbf{D}:=\frac{1}{2}[\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T], \tag{11}$$

and $\boldsymbol{\omega}$ is the angular velocity of $\boldsymbol{n}$ relative to the observer locally rotating with the fluid, which is formally defined by

$$\boldsymbol{\omega}:=\boldsymbol{w}-\boldsymbol{\Omega}, \tag{12}$$

where

$$\boldsymbol{\Omega}:=\frac{1}{2}\operatorname{curl}\boldsymbol{v}$$

is the vorticity vector. Similarly, the torque $\boldsymbol{k}$ in Eq. (3) is composed of the *elastic* torque $\boldsymbol{k}^{(e)}$ and the *viscous* torque $\boldsymbol{k}^{(v)}$, which in the Ericksen-Leslie theory are given the form

$$\boldsymbol{k}^{(e)}=\boldsymbol{n}\times\left[\operatorname{div}\left(\frac{\partial W}{\partial\nabla\boldsymbol{n}}\right)-\frac{\partial W}{\partial\boldsymbol{n}}\right], \tag{13}$$

$$\boldsymbol{k}^{(v)}=-\gamma_1\boldsymbol{\omega}-\gamma_2\boldsymbol{n}\times\mathbf{D}\boldsymbol{n}, \tag{14}$$

where

$$\gamma_1:=\alpha_3-\alpha_2$$

is the *rotational viscosity* and

$$\gamma_2:=\alpha_6-\alpha_5.$$

Finally, the energy $\mathcal{D}$ dissipated per unit volume by both viscous stresses and viscous torques can be expressed as

$$\begin{aligned}\mathcal{D}=&\,\alpha_1(\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})^2+2\gamma_2\mathbf{D}\boldsymbol{n}\cdot\boldsymbol{\omega}\times\boldsymbol{n}+\alpha_4\mathbf{D}\cdot\mathbf{D}\\&+(\alpha_5+\alpha_6)\mathbf{D}\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n}+\gamma_1\boldsymbol{\omega}\cdot\boldsymbol{\omega},\end{aligned} \tag{15}$$

where also the Parodi relation

$$\gamma_2=\alpha_3+\alpha_2 \tag{16}$$

has been used.

IV. INCIPIENT DYNAMICS

We now apply the theory just recalled to determine the incipient flow in the boundary layer described in Sec. II. Let the supporting plate be at $z=0$ in the Cartesian frame where the unit vectors along the x, y, and z axes are denoted, respectively, by $\boldsymbol{e}$, $\boldsymbol{e}_\perp$, and $\boldsymbol{\nu}$. This plate is described as a plane because here we envisage the ideal situation where no roughness is present at the molecular scale.

The velocity profile is chosen in the form

$$\boldsymbol{v}=\varphi(z)V\boldsymbol{e}, \tag{17}$$

where V is a scalar parameter with the dimensions of a velocity and φ is a function that satisfies the *no-slip* condition

$$\varphi(0)=0.$$

Both V and φ are to be determined. Requiring the latter to satisfy

$$0\leqslant\varphi(z)\leqslant1,$$

we can interpret $|V|$ as the *maximum velocity* in the whole profile.

With $\boldsymbol{v}$ as in Eq. (17), $\mathbf{D}$ and $\boldsymbol{\Omega}$ become

$$\mathbf{D}=\frac{1}{2}\varphi' V(\boldsymbol{e}\otimes\boldsymbol{\nu}+\boldsymbol{\nu}\otimes\boldsymbol{e}),$$

$$\boldsymbol{\Omega}=\varphi'\frac{V}{2}\boldsymbol{e}_\perp,$$

where a prime denotes differentiation with respect to z. We further take $\boldsymbol{n}$ as

$$\boldsymbol{n}=\cos\vartheta\,\boldsymbol{\nu}+\sin\vartheta\,\boldsymbol{e}, \tag{18}$$

where ϑ is a function of z only. Here $\boldsymbol{n}_\perp$ is given by

$$\boldsymbol{n}_\perp=\boldsymbol{e}_\perp\times\boldsymbol{n}=\cos\vartheta\,\boldsymbol{e}-\sin\vartheta\,\boldsymbol{\nu},$$

and so Eq. (7) becomes

$$\mathbf{K}_s=H_s\boldsymbol{e}_\perp,$$

where H_s is to be regarded as a function of both ϑ and z, rapidly decreasing to zero for $z>h$.

Here for W we take the one-constant approximation to Frank's elastic free energy density

$$W=\frac{1}{2}K|\nabla\boldsymbol{n}|^2,$$

where K is a positive elastic modulus. Thus Eqs. (9) and (13) deliver the following expressions for $\mathbf{T}^{(e)}$ and $\boldsymbol{k}^{(e)}$:

$$\mathbf{T}^{(e)} = -p\mathbf{I} - K(\nabla \boldsymbol{n})^T(\nabla \boldsymbol{n}) \tag{19}$$

and

$$\boldsymbol{k}^{(e)} = K\boldsymbol{n} \times \nabla^2 \boldsymbol{n}. \tag{20}$$

For $\boldsymbol{n}$ as in Eq. (18)

$$\nabla \boldsymbol{n} = \vartheta'(-\sin\vartheta\ \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \cos\vartheta\ \boldsymbol{e} \otimes \boldsymbol{\nu}). \tag{21}$$

Similarly, Eqs. (20) and (14) become

$$\boldsymbol{k}^{(e)} = K\vartheta'' \boldsymbol{e}_\perp$$

and

$$\boldsymbol{k}^{(v)} = -\gamma_1 \boldsymbol{w} + u(\gamma_1 - \gamma_2 \cos 2\vartheta)\mathbf{e}_\perp ,$$

where we have set

$$u := \varphi' \frac{V}{2}. \tag{22}$$

Thus, from Eq. (3) we read

$$\boldsymbol{w} = w\, \boldsymbol{e}_\perp ,$$

with

$$w := \frac{1}{\gamma_1}\{K\vartheta'' + H_s + u(\gamma_1 - \gamma_2 \cos 2\vartheta)\}. \tag{23}$$

On the other hand, since both $\boldsymbol{n}$ and $\boldsymbol{v}$ depend only on z, Eq. (2) becomes

$$\operatorname{div} \mathbf{T} = \frac{\partial}{\partial z}\mathbf{T}\boldsymbol{\nu} = \mathbf{0}$$

and so

$$\mathbf{T}\boldsymbol{\nu} = \boldsymbol{c}, \tag{24}$$

where $\boldsymbol{c}$ is a constant vector. This says that the traction is the same on each section parallel to the supporting plate.

Let c_e be the component of $\boldsymbol{c}$ along $\boldsymbol{e}$. Using Eqs. (19), (11), and (21)–(23), we obtain from Eq. (24) that

$$\boldsymbol{e} \cdot \mathbf{T}\boldsymbol{\nu} = f(\vartheta)u + (\alpha_2 \cos^2\vartheta - \alpha_3 \sin^2\vartheta)w = c_e , \tag{25}$$

where

$$\begin{aligned} f(\vartheta) := {} & 2\alpha_1 \sin^2\vartheta \cos^2\vartheta + \alpha_4 \\ & + (\alpha_5 - \alpha_2)\cos^2\vartheta + (\alpha_6 + \alpha_3)\sin^2\vartheta. \end{aligned}$$

We now assume that both u and w vanish in the limit as $z \to \infty$. Since there $H_s = 0$, this assumption amounts to requiring that

$$\lim_{z\to\infty} \varphi'(z) = 0, \quad \lim_{z\to\infty} \vartheta''(z) = 0. \tag{26}$$

The incipient motion is completely determined by Eqs. (23) and (25) with $c_e = 0$. Using Eq. (16), the solution to this linear system can be given the form

$$w = \frac{1}{\gamma_s}(K\vartheta'' + H_s), \tag{27}$$

$$u = \left(\frac{\gamma_1}{\gamma_s} - 1\right)\frac{K\vartheta'' + H_s}{\gamma_1 - \gamma_2 \cos 2\vartheta}, \tag{28}$$

where

$$\gamma_s(\vartheta) := \gamma_1 - \frac{2(\alpha_2 \cos^2\vartheta - \alpha_3 \sin^2\vartheta)^2}{f(\vartheta)} \tag{29}$$

is an *effective* viscosity.

Though the most interesting consequences of Eqs. (27) and (28) will be drawn below for the boundary layer where the surface effects are diluted, they are valid in the whole region occupied by the liquid crystal. In particular, the former equation tells us that here the *backflow* can easily be described as if *no flow* were present provided the rotational viscosity γ_1 is replaced by γ_s, which depends on all other viscosity coefficients besides the orientation of $\boldsymbol{n}$.

We further show that the effective viscosity γ_s obeys the inequalities

$$0 < \gamma_s \leq \gamma_1 \tag{30}$$

and so the backflow actually results in an effective reduction of the rotational viscosity. To prove Eq. (30), it suffices to insert in Eq. (15) the fields $\boldsymbol{n}$ and $\boldsymbol{v}$ as in Eqs. (18) and (17), respectively; we thus obtain

$$\mathcal{D} = 2f(\vartheta)u^2 + 4(\alpha_2 \cos^2\vartheta - \alpha_3 \sin^2\vartheta)uw + \gamma_1 w^2. \tag{31}$$

It is easily seen that this quadratic form is positive definite for all u and w provided

$$f(\vartheta) > 0, \tag{32a}$$

$$f(\vartheta)\gamma_1 - 2(\alpha_2 \cos^2\vartheta - \alpha_3 \sin^2\vartheta)^2 > 0 \tag{32b}$$

for all $\vartheta \in [0,\pi]$. Using Eq. (29), these inequalities imply both those in Eq. (30). They were already derived by Leslie: inequality (32a) readily follows from a similar analysis performed in [7], while (32b) was first proved in [8]. Though these inequalities have been exploited in several papers in the late 1970s (see, e.g., [9] and [10]), they do not seem to be widely known.

The effective viscosity γ_s equals γ_1 only if ϑ reaches the angle ϑ_a of *flow alignment*, which is defined by

$$\alpha_2 \cos^2\vartheta_a - \alpha_3 \sin^2\vartheta_a = 0$$

and exists provided α_2 and α_3 have the same sign [5]. Since, by virtue of the Parodi relation (16),

$$-2(\alpha_2 \cos^2\vartheta - \alpha_3 \sin^2\vartheta) = \gamma_1 - \gamma_2 \cos 2\vartheta,$$

use of Eq. (29) in Eq. (28) leads us to conclude that $u = 0$ for $\vartheta = \vartheta_a$. Thus, if ϑ_a exists and ϑ crosses this special value, then the slope of the velocity profile changes sign, though the sign of the resulting torque remains the same.

V. LAYER STRUCTURE

Though we are mainly interested in dynamics, we also need to know the static structure of the boundary layer through which the orienting effect of the surface is transferred to the liquid crystal in the bulk. Formally, the static solutions are obtained by setting both u and w equal to zero in Eqs. (27) and (28). Instead of finding a specific solution to these equations, we rather describe the general qualitative features that all solutions must possess.

In the usual treatment of surfaces, the director is just defined mathematically as a unit vector field assigned to a surface. For $\boldsymbol{n}$ as in Eq. (18), this would amount to defining a single angle ϑ_s. Since here, however, we have imagined that the surface field is diluted in a boundary layer, one may question what is to play the role of the surface director. We now show that for whatever dilution law of the surface field, the director orientation is almost uniform within the boundary layer.

Our basic argument relies on the fact that just because of the dilution in the bulk of the surface potential, there is no localized *surface* anchoring on the plane of contact between the liquid crystal and the active plate. Thus no localized surface torque is transmitted through this plane of contact either and so in our model the normal gradient of the director must vanish:

$$\vartheta'(0)=0.$$

This implies the existence of a layer within which the director is uniform, though its thickness might well be different from h. The variation of ϑ within this layer is governed by the contact curvature $\vartheta''(0)$. From Eq. (27) we see that

$$\vartheta''(0)=-\frac{H_s}{K}\sim\frac{1}{Lh},$$

independently of the specific dilution profile and for all reasonable dilution laws for the surface field. At the distance h from the boundary where the surface field is supposed to be confined the variation of ϑ is expected to be

$$\Delta\vartheta\simeq\frac{1}{2}\vartheta''(0)h^2\sim\frac{h}{L}.$$

Since $h/L\ll 1$, we can think of ϑ as almost constant precisely within the boundary layer where the surface field is diluted. Thus *all* static solutions must exhibit a quasi-uniform orientation within this boundary layer and so in our model it corresponds to the usual two-dimensional surface orientation.

VI. LAYER DYNAMICS

We now turn our attention to the dynamics of the boundary layer. As above, we will not solve a specific problem, which would imply specifying the dilution law; we rather attempt to derive the qualitative features common to all solutions. To make sure that our model for a diluted surface orientation is of any use, we must show that the almost uniform orientation in the boundary layer found in statics does not get destroyed by the dynamics.

Now both w and u are different from zero and possibly vary inside the boundary layer. Since the surface relaxation might be quite fast, we imagine a class of problems where we start from an initial configuration that has been perturbed by the action of an external field. Physically, it is reasonable to assume that this field is *uniform* across the boundary layer with thickness h. It might additionally be perturbed for instance by a dynamical disturbance. We instantaneously remove the field and we try to determine the incipient motion in this layer. The torque density $K\vartheta''+H_s$ in Eq. (27) is now unbalanced, but directly opposed to the torque density of the removed field. We discuss the incipient dynamics of the boundary layer. Because the field was independent of z and ϑ is almost constant within this layer, w is also so.

We can thus treat this layer as an individual whole and integrate both Eqs. (27) and (28) throughout it. We first obtain

$$h\gamma_s(\vartheta_s)w=K\vartheta_s'+\mathcal{H}_s, \tag{33}$$

where $\vartheta_s':=\vartheta'(h)$ and $K\vartheta_s'$ is the elastic surface torque transmitted through the interface between the bulk and the surface layer, while

$$\mathcal{H}_s:=\int_0^\infty H_s dz\simeq\int_0^h H_s dz$$

is the usual surface torque exerted by the surface layer.

The left-hand side of Eq. (33) can also be written as

$$h\gamma_s(\vartheta_s)\dot{\vartheta}_s,$$

where $h\gamma_s$ is the *surface viscosity*. Equation (33) was already introduced phenomenologically in [1] and [2]: It is now being justified by this model. Here the phenomenological length ℓ is shown to be equal to the extension length h of the surface potential and the viscosity η is just the effective viscosity γ_s introduced above.

For the boundary layer we can now introduce a *surface* relaxation time τ_s that can be estimated by a simple dimensional argument applied to Eq. (33):

$$\frac{1}{\tau_s}\sim\frac{K}{\gamma_s Lh}.$$

Is the real surface relaxation rate $1/\tau$ comparable to $1/\tau_s$ when we remove the applied field? The answer to this question obviously depends on whether it is possible to neglect $K\vartheta_s'$ in Eq. (33) or, which is the same, $K\vartheta''$ in Eq. (27). We consider two classes of initial director profiles, one more common than the other. In the former, the profiles are assumed to be in exact equilibrium with the applied field, whereas in the latter they are not. Both classes of profiles are expected to evolve in time towards the uniform orientation dictated by the surface field.

In the first case, the curvature is not negligible in the boundary layer: one can show that just because of the equilibrium condition there is almost a compensation on the right-hand side of Eq. (27) between curvature and surface field. The surface relaxation rate $1/\tau$ is thus found to be of the order $K/\gamma_s L^2$, which is slower than $1/\tau_s$ by the ratio h/L. This is in fact the relaxation rate of the bulk curvature created by the applied field over the coherence length $\xi_E \sim L$. The physical meaning of this compensation is just that because the surface could intrinsically relax faster, it can adjust itself quasi-adiabatically to the edge of the diffusing bulk profile.

We now imagine that the initial orientation is not exactly in equilibrium with the applied field by the action of some dynamical disturbance. We further consider the whole class of initial profiles that obey the condition $\vartheta'(0)=0$ and are, for instance, flatter than the equilibrium profile for $z\sim h$. In this case, $K\vartheta_s'$ is indeed negligible at the initial time, as is $K\vartheta_s'$ in Eq. (33), and so we can expect $1/\tau$ to be comparable to $1/\tau_s$ for the incipient motion. One can conjecture that during the short time lapse τ_s the curvature ϑ'' builds up for the profile to match the boundary orientation with the almost steady curvature in the bulk. Eventually, the director profile will look like the one in the above adiabatic regime with the slower relaxation rate $1/\tau \sim K/\gamma_s L^2$. The amplitude of the faster component in the relaxation clearly depends on the amplitude of the initial equilibrium profile.

To describe more accurately the surface relaxation, we would need to go into more analytical detail, which will form the subject of a future paper, where we shall also discuss both the direct and inverse effects of surface backflow. These effects have actually been proposed to explain the observed coupling between two surface anchoring bifurcations in thin nematic cells [11]: Also, a numerical study is presently under way [12].

VII. CONCLUSION

Using a three-dimensional model where the surface orienting field is first diluted in a thin boundary layer and then attributed to an ideal plane, we have shown that the phenomenological surface viscosity rests indeed on firm ground. In our model, the surface dissipation comes only from the director rotation and the associated backflow close to the boundary. Strictly speaking, for this model to be valid, the thickness *h* of the boundary layer should be large compared to the molecular scale *a*. Because we have assumed *L* to be larger than *h*, our model would in fact only be applicable to weak anchorings. In practice, the extension depth of the surface field could be comparable to *a*. This should make one question the validity of our detailed description of the boundary layer, but, in our opinion, integrating over the layer would still make sense and lead to the same conclusions.

Experimentally, *h* is in the range of 100 Å and so it is comparable to the coherence length of the nematic-isotropic transition, which is traditionally regarded as a macroscopic length. That *h* is indeed a macroscopic length can also be explained in other ways. There are several other sources of dissipation, which have been disregarded in our model. First, the bounding plate could be mechanically rough and so the flow would also have tangential gradients. Second, the easy orientation of the director could vary along the plate, especially if this is rough. Third, the order parameter, as well as the composition of the liquid crystal, could vary in space, and so on. All these nonuniformities should contribute to the dissipation in the boundary layer. It would be an interesting but possibly difficult task to separate experimentally each contribution: We can just say that our model for the ideal uniform ordering gives the minimum possible value for the surface viscosity. It could well be that in the experiment performed in [4] the main contribution to the surface viscosity came indeed from one of these additional mechanisms, and so this measurement of *h* would just be an upper bound for the unknown but possibly shorter extension of the surface forces in the bulk.

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